Obtaining Dispersion Curves of Damped Waves by Employing Semi Analytical Finite Element Formulation

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Abstract. Semi-analytical finite element (SAFE) method is used for modelling propagation of elastic waves in waveguides of cross-sections uniform along the propagation direction. Dispersion curves, which express the relation between the circular frequency and the wavenumber of propagating wave were obtained. Linear proportional damping of propagating waves is taken into account. The results of simulation have been compared with theoretical and experimental data available in the literature.

Keywords: Semi-analytical FEM; dispersion curve; elastic wave; complex wavenumber.

1 Introduction

Applications of guided waves have already a long history and variety of fields of usage especially for non-destructive-testing and ultrasonic measurement. Finite element methods are widely used for modelling and simulation of wave propagation. Finite element structures based on 3D elastic elements are well-suited for modelling waves in bodies of complex geometry. The waves in infinite uniform structures such as rails, bars, beams, pipes, etc., can be efficiently treated by applying the semi analytical finite element (SAFE) method. In these conditions properties of waves along the length of the uniform waveguide can be well predicted [2].

The goal of this paper is to extend the SAFE technique for obtaining dispersion relations in a damped waveguide. Dispersion curve shows the change of velocity of the wave with the change of circular frequency. SAFE methodology was first introduced by Lagasse [5] and Aalami [1]. Gavric assumed that displacement along the direction of a propagating wave is shifted by a phase of $\pi/2$ in relation with two dimensional displacement field in cross-section of the waveguide [4]. E. Viola, A. Marzani and I. Bartoli [7,8] developed this technique further introducing complex stiffness member into the model allowing modeling wave propagation in damped media. This paper explores the possibility to use a damping term in the wave equation in order to represent the wave damping phenomena.

The SAFE technique combines the analytical solution of propagating wave along the length of the uniform waveguide with the numerical solution of 3D displacement field over the cross-section of the waveguide. SAFE has an advantage compared with conventional 3D FEM approach as it offers solutions at lower computational costs. Simultaneously it enables the modeling of very short waves, since the polynomial approximation of the displacement field along the length of the waveguide is avoided [8]. Any geometrical shape of the cross section is allowed as long as the shape remains constant along the length of the waveguide. On the contrary, pure analytical solutions are feasible just for specific geometrically simple shapes of the cross-section.

2 Derivation the governing equation

Consider the elastic wave in isotropic homogenous media. We say elastic in the sense that Hooke's law for strain-stress relationship holds. The SAFE structural dynamic equation for the propagating wave reads as

$$[M]\{\ddot{U}\}+[C]\{\dot{U}\}+[K]\{U\}=0\,, \tag{1}$$

where $[M]$, $[C]$, $[K]$ are the mass, damping and stiffness matrices correspondingly. $\{U\}$ is the nodal complex displacement vector of a harmonic wave as:

$$\{U\}=\{\hat{U}\}e^{i(kz-\omega t)}\,, \tag{2}$$

where $\{\hat{U}\}$ is the real vector of amplitudes, k is the wave number (a spatial wave characteristic having a measure unit of rad/m), ω is the angular frequency (a temporal wave characteristic, having a measure unit of rad/s), t represents time, i – imaginary unit, and z is the coordinate along the direction propagation the wave.
The first and the second time derivatives of displacements R:

$$\{\dot{U}\}=\{\hat{U}\}(-i\omega)e^{i(kz-\omega t)}\,,\ \{\ddot{U}\}=\{\hat{U}\}(-\omega^2)e^{i(kz-\omega t)}\,. \tag{3}$$

Eq.(1) now can be rewritten as

$$\left([M](-\omega^2)+[C](-i\omega)+[K]\right)\{\hat{U}\}e^{i(kz-\omega t)}=0\,. \tag{4}$$

At any point of the waveguide displacements $\{U\}$, strains $\{\varepsilon\}$ and stresses $\{\sigma\}$ and the relations between them can be expressed in Cartesian coordinates as:

$$\{U\}=\begin{Bmatrix}u\\ v\\ w\end{Bmatrix},\ \{\varepsilon\}=\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{xy}\\ \varepsilon_{xz}\\ \varepsilon_{yz}\end{Bmatrix}=\begin{Bmatrix}\partial u/\partial x\\ \partial v/\partial y\\ \partial w/\partial z\\ \partial u/\partial y+\partial v/\partial x\\ \partial u/\partial z+\partial w/\partial x\\ \partial v/\partial z+\partial w/\partial y\end{Bmatrix},\ \{\sigma\}=[D]\{\varepsilon\}\ , \tag{5}$$

where $[D] = E/(1+\nu)\begin{bmatrix} \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix}$

with E being Young's modulus and ν Poisson's ratio. Expression for strains can be presented as:

$$,\{\varepsilon\} = \left[[L_x]\frac{\partial}{\partial x} + [L_y]\frac{\partial}{\partial y} + [L_z]\frac{\partial}{\partial z} \right]\{U\} \tag{6}$$

where

$$[L_x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad [L_y] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad [L_z] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

In SAFE the finite element discretization is carried out only over the cross-section of the waveguide, therefore element shape functions of two variables $a(x,y)$, $b(x,y)$ and $c(x,y)$ describing the distribution of the amplitudes over the cross-section are introduced, [8].

The scheme of the finite element discretization of the rectangular cross-section of the waveguide is shown in Fig. 1. We employ four-node first order Serendipity element, the nodal displacements of which are presented as:

$$\{\hat{U}\}_e = \begin{Bmatrix} \sum_{k=1}^{4} N_k(x,y)a_k \\ \sum_{k=1}^{4} N_k(x,y)b_k \\ \sum_{k=1}^{4} N_k(x,y)c_k \end{Bmatrix}_e e^{i(kz-\omega t)} = [N(x,y)]\{d_e\}e^{i(kz-\omega t)} \tag{8}$$

where (a_k, b_k, c_k) are the displacements of k –th node along Ox, Oy and Oz directions and $N_k(x,y)$ is the shape function of k –th node,

$\{d_e\} = \{a_1 \quad b_1 \quad c_1 \quad \cdots \quad a_4 \quad b_4 \quad c_4\}^T$ presents all displacements of all nodes of the element and matrix $[N(x, y)]$ contains all shape functions:

$$[N(x,y)] = [[NN_1(x,y)][NN_2(x,y)][NN_3(x,y)][NN_4(x,y)]] \tag{9}$$

where $[NN_j(x,y)] = \begin{bmatrix} N_j(x,y) & 0 & 0 \\ 0 & N_j(x,y) & 0 \\ 0 & 0 & N_j(x,y) \end{bmatrix}$ with $j = \overline{1,4}$.

The exponential term in (4) eq. represents the harmonic wave displacement in time and along the propagation axis. The solution is obtained by calculating all nodal displacement a, b, c amplitudes over the cross-section and one unknown for choice, ω or k. If wavenumber k is chosen as a free argument, then ω as a function $\omega(k)$ is obtained. If angular frequency ω is chosen freely, the function $k(\omega)$ is to be found. The exponential term is the analytical part of the solution while amplitude vector $\{\hat{U}\}$ is found via finite element (FE) model. Combining the two terms together into one solution is the basic idea of SAFE method. $\{\hat{U}\}$ is considered as a vector of amplitudes of all finite elements in the cross-section of a waveguide.

Now the strain in the element can be expressed as

$$\begin{aligned} \{\varepsilon_e\} &= \left[[L_x]\frac{\partial}{\partial x} + [L_y]\frac{\partial}{\partial y} + [L_z]\frac{\partial}{\partial z}\right][N(x,y)]\{d_e\}e^{i(kz-\omega t)} = \\ &= \left[[L_x][N(x,y)'_x] + [L_y][N(x,y)'_y] + ik[L_z][N(x,y)]\right]\{d_e\}e^{i(kz-\omega t)}. \end{aligned} \tag{10}$$

By introducing matrices $[B_1] = [L_x][N(x,y)'_x] + [L_y][N(x,y)'_y]$ and $[B_2] = [L_z][N(x,y)]$, relation (10) is rewritten as

$$\{\varepsilon_e\} = [[B_1] + ik[B_2]]\{d_e\}e^{i(kz-\omega t)}. \tag{11}$$

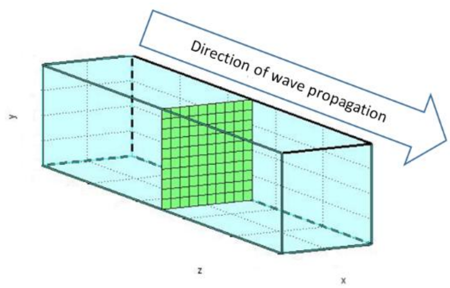

Fig. 1. Finite element representing the cross –section of the waveguide

By using Hamilton's principle it can be shown [8] that the terms in eq. (4) for a FE can be expressed as

$$[m]_e = \int_x \int_y [N(x,y)]^T \rho [N(x,y)] dxdy \ , \tag{12}$$

$$[k_1]_e = \int_x \int_y [B_1]^T [D][B_1] dxdy \ , \tag{13}$$

$$[k_2]_e = \int_x \int_y [[B_1]^T [D][B_2] - [B_2]^T [D][B_1]] dxdy \ , \tag{14}$$

$$[k_3]_e = \int_x \int_y [B_2]^T [D][B_2] dxdy \ , \tag{15}$$

where ρ is the mass density of the media. The global matrices of the elements of the cross-section are obtained by assembling the structural matrices as

$$[M] = \bigcup_{j=1}^{Nel} [m]_e \ , [K_1] = \bigcup_{j=1}^{Nel} [k_1]_e \ , [K_2] = \bigcup_{j=1}^{Nel} [k_2]_e \ , [K_3] = \bigcup_{j=1}^{Nel} [k_3]_e \tag{16}$$

where *Nel* defines the number of elements of the cross-section.

The damping term $[C]$ in eq.(1) is considered as directly proportional to the stiffness and mass matrices as:

$$[C] = \alpha [M] + \beta [K] \ , \tag{17}$$

where α and β are the proportionality coefficients. For the sake of simplicity assume that damping is proportional to the mass only and can be expressed as

$$[c]_e = \int_x \int_y [N(x,y)]^T \mu [N(x,y)] dxdy \tag{18}$$

where μ is the damping coefficient of the material. Then global damping matrix is obtained:

$$[C] = \bigcup_{j=1}^{Nel} [c]_e = \alpha [M]. \tag{19}$$

The final form of eq. (4) reads as

$$[-[M](\omega^2) + [C](-i\omega) + [K_1] + [K_2]ik + [K_3]k^2]\{\hat{U}\} e^{i(kz - \omega t)} = 0 \,. \tag{20}$$

Finding a non-zero solution to this equation leads to corresponds to the generalized eigenvalue problem as

$$\det\left(-[M](\omega^2) + [C](-i\omega) + [K_1] + [K_2]ik + [K_3]k^2\right) = 0 \,. \tag{21}$$

It is required to solve the eigenvalue problem of dimensionality equal to total number degrees of freedom of the cross-section of the waveguide. The eigenvalue problem is solved either with a given ω, or with a given k. In case the wavenumber k is chosen freely, the eigenvalue problem is simplified as:

$$\left[-[M](\omega^2)+[C](-i\omega)+[K(k)]\right]\left\{\hat{U}\right\}=0 \tag{22}$$

where $K(k)$ matrix contains complex numbers and its value depends on wavenumber k.

As complex value eigenproblem is to be solved, its dimension doubles in order to encompass the real and the complex parts. By using notation:

$$[A]_1=\begin{bmatrix} 0 & [K] \\ [K] & [C](-i) \end{bmatrix}, [B]_1=\begin{bmatrix} [K] & 0 \\ 0 & [M] \end{bmatrix}, \{Q\}=\begin{bmatrix} \left\{\hat{U}\right\}/\omega \\ \left\{\hat{U}\right\} \end{bmatrix} \quad , \tag{23}$$

eq. (23) can be rewritten as:

$$\left[[A]_1-[B]_1\omega\right]\{Q\}=0\,. \tag{24}$$

In case the angular frequency ω is chosen freely, the eigenproblem is simplified using $[A]_2$ and $[B]_2$ matrices and vector $\{Q\}$ to another form as:

$$\left[[A]_2-[B]_2 k\right]\{Q\}=0 \tag{25}$$

$$[A]_2=\begin{bmatrix} 0 & [K_1]-[M]\omega^2-[C](i\omega) \\ [K_1]-[M]\omega^2-[C](i\omega) & [K_2] \end{bmatrix} ,$$

$$[B]_2=\begin{bmatrix} [K_1]-[M]\omega^2-[C](i\omega) & 0 \\ 0 & -[K_3] \end{bmatrix}, \{Q\}=\begin{bmatrix} \left\{\hat{U}\right\} \\ \left\{\hat{U}\right\}k \end{bmatrix}. \tag{26}$$

3 Results

Consider a waveguide of rectangular cross-section 0.01 × 0.01 m. The material is aluminum with density $\rho=2700$ kg/m^3, Young's modulus $E=6.9\cdot 10^9$ Pa and Poisson's ratio $\nu=0.32$. We investigate the convergence of angular frequency value as the mesh refinement over the cross-section is increased at selected constant wave number of 1 rad/m. The results for first five modes are shown below (Fig.2).

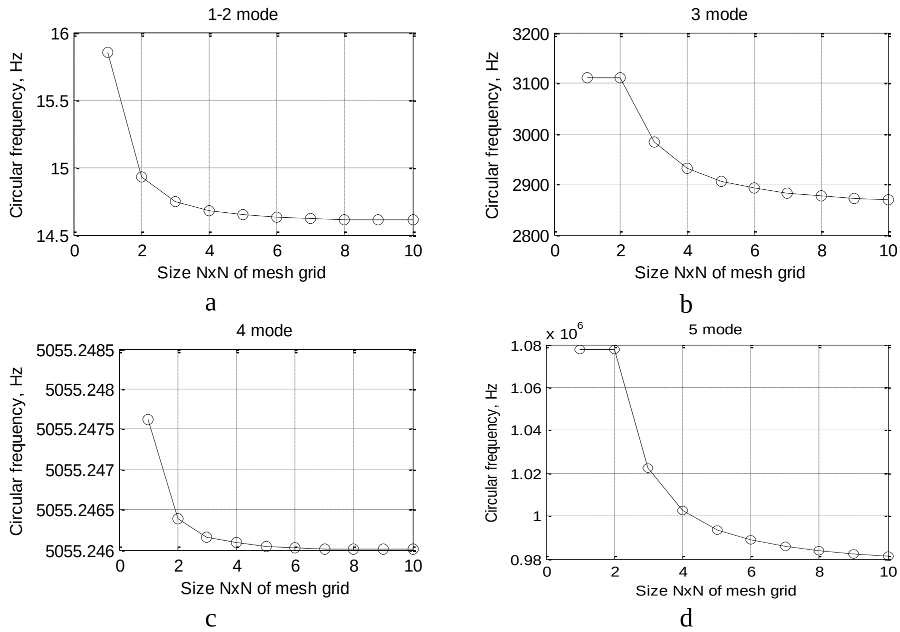

Fig. 2. ω convergence of the wave angular frequencies values as mesh refinement is increased at wavenumber $k=1$ rad/m a) first and second modes b) third mode c) fourth mode d) fifth mode.

Consider the same waveguide , the cross-section mesh 10×10 in the undamped case ($[C]=0$ in eq. (22)). In ω versus k scenario eigenproblem (22) becomes

$$\left[-[M](\omega^2)+[K]\right]\left\{\hat{U}\right\}=0\,. \tag{27}$$

There are $tdof=11\cdot 11\cdot 3=363$ solutions for ω, which are real numbers. Dispersion curves are obtained as in Fig.3a. For a comparison a dispersion curve extracted from the single_cross-section element model of the same waveguide (1×1 mesh) having $tdof=2\cdot 2\cdot 3=12$ is presented in Fig. 3b. $F0, F1$ indicate first and second flexular modes, $T0$ - first torsional mode, $L0$ - first longitudinal mode.

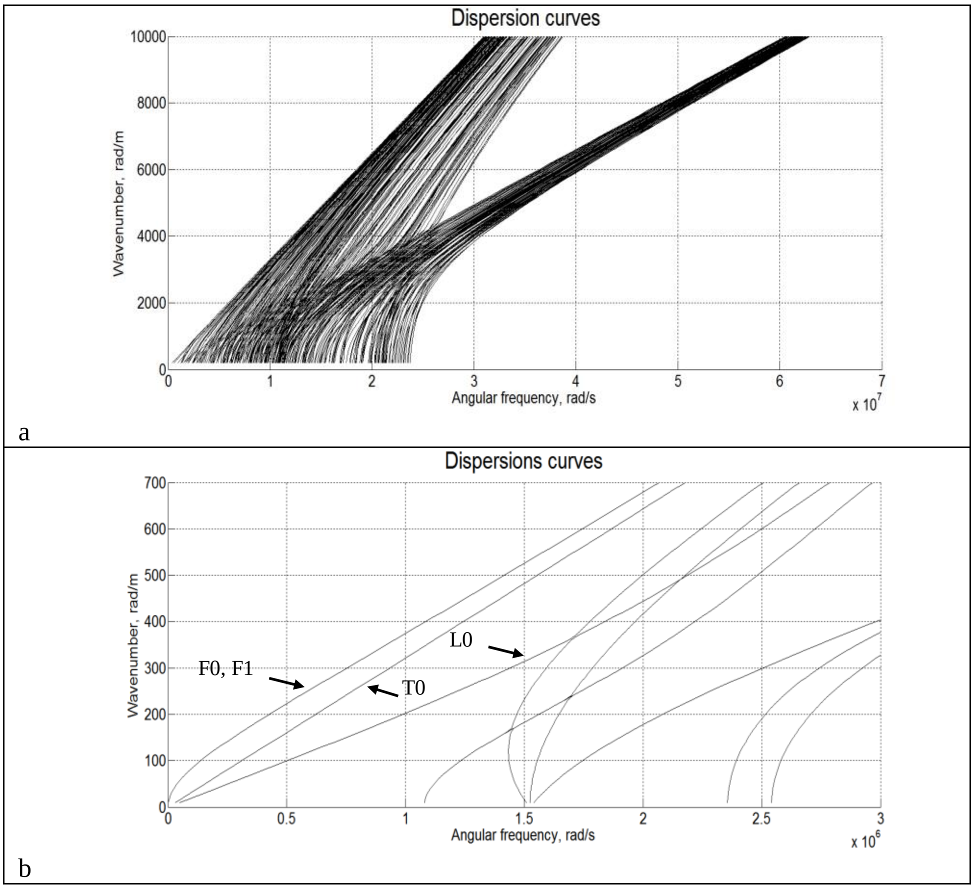

Fig. 3. Dispersion curves at cross-section discretization 10×10 (a) and single FE cross-section 1×1(b)

Consider the same waveguide, where the SAFE model of which formulated as (26). This means that the wave numbers are treated as unknowns while the values of circular frequencies are selected freely. Here we investigate the impact of damping on the obtained dispersion curves. In eigenproblem (26) 2×*tdof* complex solutions for wavenumbers are obtained. We are interested only in solutions expressed as $k = \pm(a+bi)$, while the solutions of the form $k = \pm(a-bi)$ are discarded for they are meaningless because the solutions of this form would mean increasing amplitudes of propagating wave (here a and b are scalars of the same signs). The imaginary part of the wavenumber describes the spatial attenuation of the wave envelope. The results with 1×1 single FE model grid in cross-section of a waveguide are shown in (Fig. 4) as different values of the coefficient α in the damping term $[C] = \alpha[M]$ are used. The results are presented as angular frequency against the real part of the wavenumber. It can be seen that coefficient α has to take rather large values in order to exhibit a tangible effect on the dispersion curve.

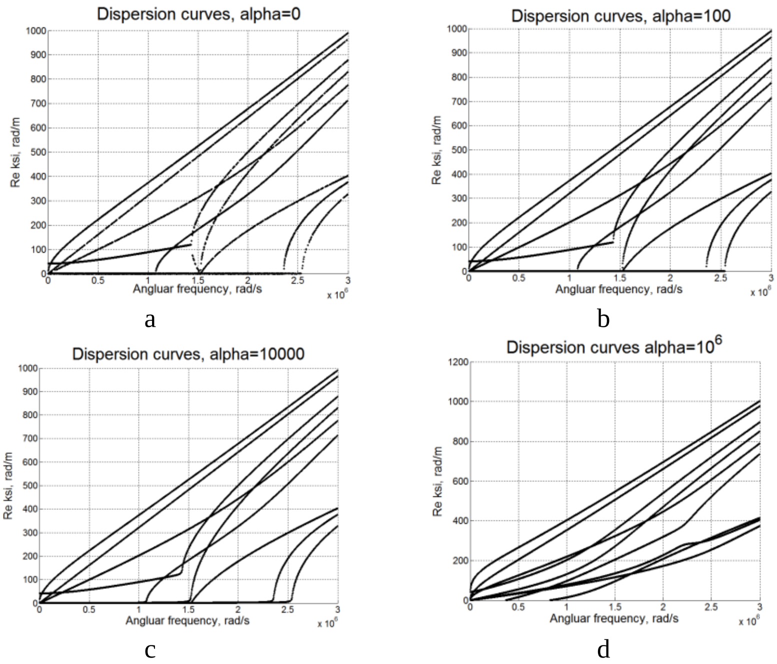

Fig. 4. Dispersion curves in case of damped waveguide: a) $\alpha = 0$, b) $\alpha = 100$, c) $\alpha = 10^4$, d) $\alpha = 10^6$

Solving (26) eigenproblem provides more physically feasible solutions when compared to (25) equation. Fig.5 displays the results obtained by using *3×3* mesh cross-section in the waveguide. Dependency $\omega(k)$ provides less solutions than $k(\omega)$. In the undamped case solutions only for propagating modes are extracted from (25) eigen problem while (26) eigenproblem Therefore a deeper investigation on angular frequency versus complex wavenumbers k is required. It would be appropriate to plot ω , $\mathrm{Re}(k)$ and $\mathrm{Im}(k)$ of the solutions in 3D space. Hao Liu suggest sorting of wavenumber in three-dimensional space by curvature of dispersion curves [6]. Further research on this subject is planned in near future.

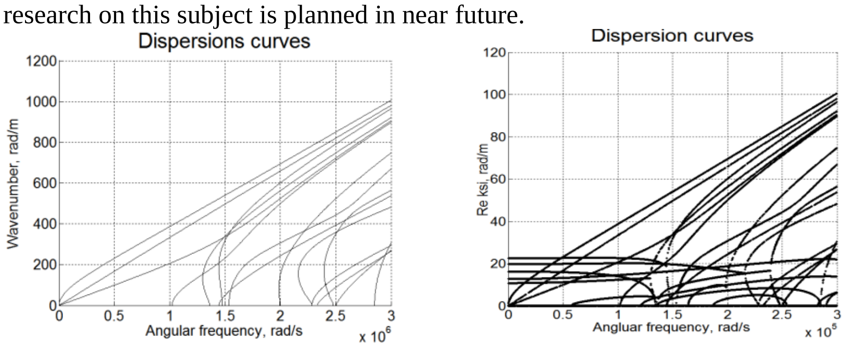

Fig. 5. Dispersion curves acquired on different dependencies: $\omega(k)$ and $k(\omega)$

For testing the validity of the waveguide model we choose to calculate the phase velocity dependence against circular frequency on the copper plate having the following properties: cross-section $2 \cdot 10^{-2} \times 2.83 \cdot 10^{-5}$ m^2, density $\rho = 8500$ kg/m^3, Young's modulus $E = 99 \cdot 10^9$ Pa, Poisson's ratio $\nu = 0.37$. The experimental measurement data considering Lamb waves in such plate were reported in [3]. Fig. 6 presents the results obtained by using 2×4 mesh over the cross-section in undamped case. The results of these computations are very close to experimental results presented in [3].

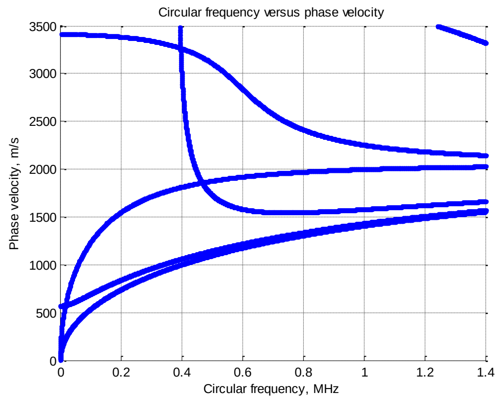

Fig. 6. Angular frequency versus phase velocity in copper plate

4 Conclusions

The SAFE method offers an efficient approach for modeling and simulation of wave propagation along uniform cross-section waveguides. The cross-section may be composed of different materials or layers, isotropic or anisotropic. Here the SAFE method was successfully used to obtain dispersion curves in undamped, as well as, proportionally damped structures, the convergence of the model and adequacy of the obtained results were demonstrated. However, deeper insight into proper sorting for wavenumbers obtained in the solution and automatic grouping of the obtained modes into proper categories is needed. The assumption that angular frequency is taken as a selected real number is similar to the analysis of forced wave propagation, discussion on which is not included in this paper. This topic is to be covered in near future. It is worth reminding that in this research we considered a very simple approach to damping, the term of which is proportional to mass matrix only.

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